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SOME CONDITIONS WHICH ALMOST CHARACTERIZE FROBENIUS GROUPS

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ABSTRACT
The main result of this paper is the following: Let G be a group with a proper non-trivial normal subgroup H such that each coset of H distinct from H is contained in a conjugacy class of G. If G is not a Frobenius group with kernel H then one of H or G/H is a p-group. The hypothesis of this theorem is shown to be equivalent to a condition on characters of G. The only group the author knows which satisfies this hypothesis and is not either Frobenius or a p-group is one of order 72.

§1. Introduction

If G is a Frobenius group with kernel H the irreducible characters of G can be divided into two families, those with H in the kernel and those for which H is not in the kernel. Let us label the second set χ_1, χ_n. Each χ_i is induced from a character, say, ξ_i of H and χ_i, when restricted to H, is the sum of the conjugates of ξ_i, see [1]. So each χ_i vanishes on $G\backslash H$ and there exists a set of integers α_i, $i = 1, \cdots, n$ such that $\Sigma_{i=1}^{n} \alpha_i\chi_i$ is constant $(= -1)$ on $H^{\#}$. We extract these ideas as

Hypothesis (F1). G is a finite group with a proper normal subgroup $H \neq 1$ and a set of irreducible non-trivial characters of $G, \chi_1, \cdots, \chi_n$, where n is a natural number, such that

(a) χ_i vanishes on $G\backslash H$ and

(b) there exist natural numbers $\alpha_1, \cdots, \alpha_n > 0$ such that $\Sigma_{i=1}^{n} \alpha_i\chi_i$ is constant on $H^{\#}$.

Another property enjoyed by Frobenius groups is: if x is in $G\backslash H$ then xy is conjugate to x for all y in H.

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HYPOTHESIS (F2). G is a finite group with a proper normal subgroup $H \neq 1$ such that if $x \in G \backslash H$, x is conjugate to xy $\forall y \in H$.

The first result is the following:

THEOREM 1. *G satisfies hypothesis* (F1) *if and only if G satisfies hypothesis* (F2).

This result will be proved in §3. In §2 we investigate some properties of groups satisfying (F2). We prove a result which when combined with Theorem 1 leads to the following theorem:

THEOREM 2. *Let G be a group satisfying* (F2) (*or* (F1)) *then G satisfies one of the following conditions*:

(i) *G is a Frobenius group with kernel H,*

(ii) *H is a p-group for some prime p, or*

(iii) *G/H is a p-group for some prime p.*

It is not too difficult to show that if H and $[G: H]$ have coprime orders and G satisfies (F2) then G is a Frobenius group with kernel H. Thus the main argument is to force either H or G/H to be a p-group if G is not Frobenius. There are certainly p-groups G satisfying these hypotheses, e.g. extra-special groups. May I thank Ian D. Macdonald for pointing out to me that a number of simple thoughts on p-groups with this structure turned out to be false. The only example the author knows where G/H is a p-group and H is not, is when G has order 72 and is the Frobenius group which has kernel of order 9 and the complement is quaternion of order 8. We choose H to be the group of order 18. If x is any element of order 4 in G, x has 18 conjugates, and since x is conjugate to x^{-1} and Hx contains x^{-1} we can see that Hx is precisely the conjugacy class of x. One can see from the discussion in §2 that it is, unfortunately, not possible to use this argument for Frobenius groups unless the complement is a 2-group. We comment here that since Hx contains only elements of the same order as x the results of Hughes Kegel Thompson [1, V §8] play an important role.

In the situation where H is a p-group we can investigate the structure of $G/C_G(H/H_0)$ where H/H_0 is a chief factor of G. This action has the property that any non-trivial p'-element acts fixed-point-freely on H/H_0. We can also observe that $C_G(H/H_0)$ is precisely $O_p(G)$ since no p'-element of G can centralize H/H_0. Using some of the more recent classification theorems and p-modular representations of known groups W. B. Stewart [2] has been able to classify the possible modules and groups which can occur as $G/O_p(G)$. Using this result we have the following:

COROLLARY. *Let G satisfy* (F2) *and assume G is not a soluble group nor is G Frobenius. If H is a p-group then* $G/O_p(G)=\bar{G}$ *has a normal subgroup* $\bar{K}$ *such that* $\bar{G}/\bar{K}$ *is soluble and*

(a) *if* $p=2$ *then* $\bar{K}\cong \mathrm{SL}(2,2^n), Sz(2^{2m+1})$ *or* $\mathrm{SL}(2,2^n)\times Sz(2^{2mb})$,

(b) *if* $p=3$ *then* $\bar{K}\cong \mathrm{SL}(2,3^n)$, $\mathrm{SL}(2,5)$, $\mathrm{SL}(2,7)$ *or* $\mathrm{SL}(2,17)$,

(c) *if* $p\geqq 5$ *then* $\bar{K}\cong \mathrm{SL}(2,p^n)$.

If one refers to W. B. Stewart [2] then more information can be obtained both as to the nature of the possible modules on which these groups act as well as the structure of $\bar{G}/\bar{K}$.

In the case where G/H is a p-group the author has been able to make limited progress, certainly none to warrant mentioning, although there is an overlap with some work on groups of "central type". A group G has been called a group of central type if it contains an irreducible character whose degree when squared equals $[G: Z(G)]$. For some results in this topic and a good bibliography the reader is referred to R. Merris (J. Res. Nat. Bur. Standards Sect. B, **80B** (1976), 259).

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§2. Groups satisfying (F2)

In this section we consider a group G satisfying (F2). We prove a number of lemmas about such a group under this hypothesis leading up to a proof of Theorem 2.

LEMMA 1. *Let* $x\in G$ *but* $x\notin H$. *Let* $\bar{\ }$ *denote the homomorphism from G to* G/H. *Then*

$$|C_G(x)|=|C_{\bar{G}}(\bar{x})|.$$

PROOF. Since each coset of H which is not H is contained in a conjugacy class, any conjugacy class not contained in H is a union of cosets of H. If Hu and Hv are in the same conjugacy class, u and v are conjugate, in particular $\bar{u}$ and $\bar{v}$ are conjugate in $\bar{G}$. Similarly if $\bar{u}$ and $\bar{v}$ are conjugate in $\bar{G}$, Hu and Hv are in the same conjugacy class. Hence $[G: C_G(x)]=[G: C_{\bar{G}}(\bar{x})].|H|$.

Thus $|G|/|C_G(x)|=|G|/|C_{\bar{G}}(\bar{x})|$ and so $|C_G(x)|=|C_{\bar{G}}(\bar{x})|$ as required.

LEMMA 2. *If x is not in H and has order m and* $y\in C_H(x)$ *then the order of y divides m.*

PROOF. xy has order m and so $x^my^m=1$ and so $y^m=1$.

LEMMA 3. *If x is not in H and has order m modulo H and $y \in C_H(x)$ such that y has prime order p say, then p divides m, where $m \neq 1$.*

PROOF. We prove this by showing that the order of x is only divisible by primes dividing m. Assume x has order divisible by precisely p^e where $e \geqq 1$ and p does not divide m. Then $x = uv$ where u has order prime to p, v has order p^e and $[u, v] = 1$. Now $x^m = u^m v^m$ and so $v^m = x^m (u^m)^{-1}$ but $x^m \in H$. This implies that v^m and u^m have the same order or that v^m and u^m are both in H. However v^m and u^m have coprime orders (not both 1) and so v^m and u^m are in H. Now as m and p are coprime we know that $v \in H$. But $[v, u] = 1$ and v has order coprime to u contradicting Lemma 2. We can now complete the argument since if y centralizes x and y has order p, p divides the order of x by Lemma 2 and so p divides m.

LEMMA 4. *If there is an element of prime order not in H then H is nilpotent.*

PROOF. This is a direct consequence of the theorem due to Hughes Thompson Kegel [1; V §8].

PROPOSITION 1. *If H and G/H are of coprime order then G is a Frobenius group with kernel H.*

PROOF. Now we know by Lemma 3 that if $y \in H, C_G(y) \subseteq H$. Also by Lemma 4 we have that H is nilpotent. Using the result of Schur–Zassenhaus [1; I §18] we have a complement to H and now it is easy to see that G is a Frobenius group.

LEMMA 5. *If there is an element x of G which has order pq modulo H where p and q are distinct primes then H is nilpotent.*

PROOF. By Lemma 3, $C_H(x^p)$ is a q-group and $C_H(x^q)$ is a p-group. So $C_H(x^p) \cap C_H(x^q) = 1 \supseteq C_H(x)$. So $x^{pq} = 1$ since $x^{pq} \in C_H(x)$. Thus x^p is an element of prime order not in H and so by Lemma 4, H is nilpotent.

We can now turn our attention to the structure of the Sylow subgroups of G. A simple application of Lemma 1 enables us to prove

LEMMA 6. *If P is a Sylow p-subgroup of G and there is an $x \in P$ such that $x \notin H$ and $C_P(x).(H \cap P) = P$ then $P \cap H = 1$.*

PROOF. Since $|C_G(x)| = |C_{\bar{G}}(\bar{x})|$ where $\bar{\ }$ denotes the homomorphism from G to G/H, a Sylow p-group of $C_G(x)$ has order at most $[P: H \cap P]$. Since $C_{H \cap P}(x) \neq 1$ when $H \cap P \neq 1$ we have a contradiction if $C_p(x).(H \cap P) = P$.

LEMMA 7. *Let p be a prime which divides the order of H and* $[G:H]$. *Let P be a Sylow p-subgroup of G then* $P/P \cap H$ *contains an elementary Abelian subgroup of order* p^2.

PROOF. If $P/P \cap H$ does not satisfy the conclusions of the lemma then $P/P \cap H$ is cyclic or generalized quaternion [1; III §8]. If $P/P \cap H$ is cyclic and $x(P \cap H)$ is a generator for $P/P \cap H$ then $C_P(x).(P \cap H) = P$ which is false by Lemma 6. Hence we may assume that $P/P \cap H$ is a generalized quaternion group of order say 2^a.

Let $x(P \cap H)$ be an element of order 2^{a-1}. Then $x^{2a-2} = z$ is such that $z^2 \in H$ and $C_{P/P\cap H}(zH) = P/P \cap H$. Also $|C_P(z)| \geqq 2^{a-1}$ since $|\langle x \rangle| \geqq 2^{a-1}$. By Lemma 1, $|C_P(z)| = 2^a$ or 2^{a-1} but as $|C_{P\cap H}(z)| \neq 1$ we know that $|C_P(z)| = 2^a$. If $z^2 \neq 1$, $|\langle z \rangle| > 2^a$ and $C_P(z) = \langle x \rangle$ and $z^4 = 1$ with $\langle z^2 \rangle = Z(P)$. As G is an (F2)-group there exists $s \in G$ such that $x^s = xz^2$. Then $(x^2)^s = x^2$ and $(x)^{s^2} = (xz^2)^s = xz^4 = x$. Thus s is a 2-power element which centralizes z and hence is in $\langle x \rangle$. This contradicts the original definition of s.

Thus we have that $z^2 = 1$. Then every element of $(P \cap H)z$ has order 2 and z inverts $P \cap H$ which is therefore Abelian. Since $|C_{P\cap H}(z)| = 2$, $P \cap H$ now has at most one involution and so is cyclic. If $|P \cap H| > 2$, $P/C_P(P \cap H)$ is Abelian but $C_P(P \cap H) = P \cap H$ which is a contradiction. So $|P \cap H| = 2$ but now since the Schur multiplier of a quaternion group is trivial [1; V §25] we can assume z is central in P which by Lemma 6 is false.

We can now complete this section by proving

PROPOSITION 2. *Let G be a group satisfying* (F2). *Then either* (i) *G is a Frobenius group or* (ii) *H is a group of prime power order or* (iii) *G/H is a group of prime power order.*

PROOF. By Lemma 5, H is nilpotent unless all elements of G/H have prime power order. We also know that if a prime p does not divide the order of H and does divide the order of G, H is nilpotent by Lemma 4. Thus if H is not nilpotent and P is a Sylow p-subgroup of G then $P \cap H \neq 1$. Assume now that H is not nilpotent and let M/H be a minimal normal subgroup of G/H. Assume M/H is an elementary Abelian subgroup and G/H is not a group of prime power order. If p is a prime which does not divide the order of M/H but does divide $[G: H]$ then $P/P \cap H$ acts fixed-point freely on M/H, where P is a Sylow p-group of G. Hence $P/P \cap H$ has no elementary Abelian subgroup of order p^2. This contradicts Lemma 7.

If M/H is a non-Abelian subgroup we have by M. Suzuki [3] that M/H is isomorphic to one of PSL(2,5), PSL(2,7), PSL(2,9), PSL(2,17), PSL(3,4), $Sz(8)$ or $Sz(32)$. Hence G/H is one of these groups extended by a subgroup of its automorphism group. But in all cases G/H has a cyclic Sylow p-subgroup for some prime p and this is again false.

So we may now assume that H is nilpotent or $[G: H]$ is a prime power. To prove the proposition we need only show that if H satisfies neither (ii) nor (iii) then it is a Frobenius group. By Proposition 1 we can assume there is a prime p such that if P is a Sylow p-subgroup of G, neither $P/P \cap H$ nor $P \cap H$ are trivial. Let R be a Sylow subgroup of H distinct from P. Then $N_G(R) \supseteq P$ and $C_G(R) \supseteq P \cap H$. Hence $P/P \cap H$ acts fixed-point freely on R and this contradicts Lemma 7.

§3. Proof of Theorem 1

Assume that G satisfies (F1). Let $|G| = g$ and $|H| = h$. We consider the equation $(\Sigma \alpha_i \chi_i, \chi_j) = \alpha_j$.

$$\therefore \quad \alpha_j = \frac{1}{g}\Big(\sum_i \alpha_i\chi_i(1)\chi_j(1) + c\sum_{x\in H^\#} \overline{\chi_j(x)}\Big) \qquad \text{where } c = \Sigma_i \alpha_i \chi_i(x),$$

$$\therefore \quad g\alpha_j = \Big(\sum_i \alpha_i\chi_i(1)\Big)\chi_j(1) - c\chi_j(1) + c\sum_{x\in G}\overline{\chi_j(x)},$$

$$\text{(1)} \qquad \therefore \quad g\alpha_j = \chi_j(1)\Big(\Big(\sum_i \alpha_i\chi_i(1)\Big) - c\Big),$$

$$\therefore \quad g = (\chi_j(1)/\alpha_j)\Big\{\sum_i \alpha_i\chi_i(1) - c\Big\}.$$

$$\therefore \quad 0 = (\chi_j(1)/\alpha_j - \chi_k(1)/\alpha_k).\Big(\Big\{\sum_i \alpha_i\chi_i(1)\Big\} - c\Big)$$

for all pairs $1 \leqq j \neq k \leqq n$.

Since $g \neq 0$ we have that

$$\text{(2)} \qquad \chi_j(1)/\alpha_j = \chi_k(1)/\alpha_k = f \qquad \text{say } \forall j, k.$$

Consider the equation $(\Sigma_i \alpha_i \chi_i, 1_G) = 0$,

$$\text{(3)} \qquad \therefore \quad 0 = \Big(\sum_i \alpha_i\chi_i(1)\Big) + (h-1)c.$$

Combining (1), (2) and (3) we obtain

$$g = f\left(\left(\sum_i \alpha_i^2 f\right) + \left(\sum_i \alpha_i^2 f\right)/(h-1)\right).$$

If we put $\Sigma\alpha_i^2 = m$ we have

$$g = mf^2(1 + 1/(h-1))$$

$$\therefore \quad g = mf^2 h/(h-1).$$

Let $\theta_1 \cdots \theta_r$ be the remaining irreducible characters of G. Then

$$\sum_{i=1}^{n} \chi_i(1)^2 + \sum_{i=1}^{r} \theta_i^2(1) = g,$$

$$\sum_{i=1}^{r} \theta_i^2(1) = g - \sum_{i=1}^{n} \alpha_i^2 . (g(h-1))/(mh),$$

$$\sum_{i=1}^{r} \theta_i^2(1) = g - \frac{g(h-1)}{mh} . m = g/h.$$

Now since H is normal in G there exist characters of G which contain H in their kernel and the sum of the squares of their degrees equals g/h. Hence these plus the $\chi_1, \cdots, \chi_n$ are all the characters of G. Now let $x \in G \backslash H$ and let $y \in H$. Then xy is not in H. For a character χ_i, $\chi_i(x) = \chi_i(xy) = 0$. For a character θ_i, $\theta_i(x) = \theta_i(xH) = \theta_i(xy)$. Thus x and xy have the same character values for all the characters of G and hence are conjugate [1].

We now assume that G satisfies (F2). Let $\theta_1, \cdots, \theta_r$ be the irreducible characters of G with H in the kernel and $\chi_1, \cdots, \chi_n$ be the remaining characters. Thus $|C_G(x)| = \Sigma_{i=1}^r |\theta_i(x)|^2 + \Sigma_{i=1}^n |\chi_i(x)|^2$. Also

$$|C_{G/H}(xH)| = \sum_{i=1}^{r} |\theta_i(xH)|^2 = \sum_{i=1}^{r} |\theta_i(x)|^2 = |C_G(x)|$$

by Lemma 1. So $\Sigma_{i=1}^n |\chi_i(x)|^2 = 0$ and thus $\chi_i(x) = 0$, $i = 1, \cdots, n$ and this is true for all x not in H.

Consider the character χ_i for some $i, 1 \leqq i \leqq n$. There exists an irreducible character ξ_i of H and a natural number e_i such that $\chi_i|_H = e_i \Sigma \xi_i^u$ where u runs over a system of coset representatives for the stabilizer of ξ_i. If $j \neq i$ the irreducible characters of H involved in $\chi_j|_H$ are disjoint from those involved with χ_i for otherwise $\chi_j|_H$ would be a multiple of $\chi_i|_H$. But since χ_i and χ_j vanish outside H, χ_j would be a multiple of χ_i and that is false. Let η be any non-trivial irreducible character of H. Then

$$(\eta^G, \theta_i)_G = (\eta, \theta_i|_H)_H = (\eta, \theta_i(1), 1_H) = 0, \qquad i = 1, 2, \cdots, r.$$

Thus η^G involves a character of the form χ_i.

Let $e = \text{l.c.m.}\ (e_i: 1 \leqq i \leqq n)$. So we have

$$\left(\sum_{i=1}^{n} (e/e_i)\xi_i(1)\chi_i \right)_H = e\rho_H - e1_H$$

where ρ_H is the regular character of H. Hence with $\alpha_i = (e/e_i)\xi_i(1)$

$$\sum_{i=1}^{n} \alpha_i\chi_i \qquad \text{is constant on } H.$$

Thus G satisfies (F1) and this completes the proof of Theorem 1.

References

1. B. Huppert, *Endliche Gruppen I*, Springer-Verlag, Berlin–Heidelberg–New York, 1967.
2. W. B. Stewart, *Largely fixed-point-free groups*, to appear.
3. M. Suzuki, *On a class of doubly transitive groups*, Ann. of Math. (2) **75** (1962), 105–145.

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